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# The relativistic Coulomb problem for particles with arbitrary half-integer spin

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**Abstract**

Using relativistic tensor-bispinorial equations proposed in Niederle and Nikitin (2001 *Phys. Rev.* D **64** 125013) we solve the Kepler problem for a charged particle with arbitrary half-integer spin interacting with the Coulomb potential.

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## 1. Introduction

Exactly solvable problems in quantum mechanics are quite important and illustrative but rather rare (see, e.g., [2]). They can be described fully and in a straightforward way free of various complications caused by the perturbation method. The very existence of exact solutions of these problems is usually connected with their non-trivial symmetries which are mostly of particular interest by themselves. In addition exact solutions form complete sets of functions which can be used to find solutions of other problems.

One of the triumphs of the Dirac theory for the electron is that it solves the problem of electron motion in the Coulomb potential. The exact Sommerfeld formula for the related energy levels is one of the cornerstones of relativistic quantum theory. However, extension of these results to the case of charge particles with higher spin appeared to be very complicated since even in the case of spin $s = 1$ the corresponding relativistic wave equation (RWE) (namely, the Kemmer–Duffin equation) is not free of inconsistencies and predicts the orbital particle will fall down on the attracting centre [3, 4]; a contemporary treatment of this problem can be found in [5] . The other well-known problems of RWE for particles with higher spins are: violation of causality [6], ill-defined interaction with a constant and homogeneous external magnetic field [7] and so on (for details see [1, 8]).

In paper [1] relativistic wave equations for particles of arbitrary half-integer spin were proposed which are free of inconsistencies typical for other wave equations for higher spins. They are causal and allow the correct value of the gyromagnetic ratio $g = 2$. In addition these equations have well-defined quasi-relativistic limits which admit a good physical interpretation and describe the Pauli, spin–orbit and Darwin couplings.

In the present paper we apply the tensor-spinorial equations [1] to solve the Coulomb problem for particles with *arbitrary* half-integer spin.

## 2. Relativistic wave equations for a particle with arbitrary half-integer spin

First, we briefly review the RWE for free particles with arbitrary half-integer spin $s$ and mass $m$ described in [1]. The corresponding wavefunction $\psi^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]}$ is an irreducible tensor with respect to the complete Poincaré group of rank $2n = 2s-1$ antisymmetric w.r.t. permutations of indices in the square brackets and symmetric w.r.t. permutations of pairs of indices $[\mu_i,\nu_i] \Longleftrightarrow [\mu_j,\nu_j]$, $i, j = 1, 2, \ldots, n$. The irreducibility requirement also means that convolutions w.r.t. any pair of indices and cyclic permutations of any triplet of indices of the wavefunction reduce it to zero. In addition, components of $\psi^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]}$ are bispinors of rank 1. This means that the wavefunction has an additional spinorial index $\alpha$ (which we omit) running from 1 to 4.

The equation of motion has the following form [1]:

$$\left(\gamma_\lambda p^\lambda - m\right)\psi^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]} - \frac{1}{4s}\Sigma_{\mathcal{P}}\left(\gamma^{\mu_1}\gamma^{\nu_1} - \gamma^{\nu_1}\gamma^{\mu_1}\right)p_\lambda\gamma_\sigma\psi^{[\lambda\sigma][\mu_2\nu_2]\cdots[\mu_n\nu_n]} = 0. \quad (1)$$

Here $\gamma_\nu$ are the Dirac matrices, $p^\mu = \mathrm{i}\frac{\partial}{\partial x_\mu}$ and the symbol $\Sigma_{\mathcal{P}}$ denotes the sum over all possible permutations of subindices $(2, \ldots, n)$ with 1.

In addition, the wavefunction $\psi^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]}$ has to satisfy the static constraint [1]

$$\gamma_\mu\gamma_\nu\psi^{[\mu\nu][\mu_2\nu_2]\cdots[\mu_n\nu_n]} = 0, \quad (2)$$

which is necessary to reduce the number of independent components of the tensor-spinor from $16s$ to $4(2s+1)$. In order to obtain a theoretically recognized $2(2s+1)$-component wavefunction we impose on $\psi^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]}$ additionally either Majorana condition or a parity-violating constraint $(1+\mathrm{i}\gamma_5)\psi^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]} = 0$ (or $(1-\mathrm{i}\gamma_5)\psi^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]} = 0$).

Equations (1), (2) are manifestly invariant with respect to the complete Poincaré group and admit the Lagrangian formulation. For the case of a charged particle interacting with an external electromagnetic field this equation is generalized to the following form [1]:

$$\begin{aligned}\left(\gamma_\lambda\pi^\lambda - m\right)\psi^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]} &- \frac{1}{4s}\Sigma_{\mathcal{P}}\left(\gamma^{\mu_1}\gamma^{\nu_1} - \gamma^{\nu_1}\gamma^{\mu_1}\right)\pi_\lambda\gamma_\sigma\psi^{[\lambda\sigma][\mu_2\nu_2][\mu_3\nu_3]\cdots[\mu_n\nu_n]}\\ &+ \frac{2\mathrm{i}ek}{sm}\Sigma_{\mathcal{P}}\left(\frac{1}{4}\gamma_\alpha\gamma_\sigma F^{\alpha\sigma}\psi_+^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]} + F_\alpha{}^{\mu_1}\psi_+^{[\nu_1\alpha][\nu_2\mu_2]\cdots[\nu_n\mu_n]}\right.\\ &\left. - F_\alpha{}^{\nu_1}\psi_+^{[\mu_1\alpha][\nu_2\mu_2]\cdots[\nu_n\mu_n]}\right) = 0. \end{aligned} \quad (3)$$

Here $\pi_\lambda = p_\lambda - eA_\lambda$, $A_\lambda$ and $F^{\mu\nu}$ are vector-potential and tensor of the external electromagnetic field, $k$ is an arbitrary parameter and

$$\psi_\pm^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]} = \psi^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]} \pm \frac{1}{2n}\gamma_5\Sigma_{\mathcal{P}}\varepsilon^{\mu_1\nu_1}{}_{\lambda\sigma}\psi^{[\lambda\sigma][\mu_2\nu_2][\mu_3\nu_3]\cdots[\mu_n\nu_n]},$$

so that

$$\psi^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]} = \tfrac{1}{2}\left(\psi_-^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]} + \psi_+^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]}\right).$$

Equation (3) contains both the minimal and anomalous interaction of a particle with an external field [1]. Moreover, putting $k = 2s-1$ for the anomalous coupling constant $k$ we get the 'natural' value $g = 2$ for the gyromagnetic ratio [1].

Solving equation (3) for $\psi_-^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]}$ we obtain

$$\psi_-^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]} = \frac{1}{m}\gamma_\lambda\pi^\lambda\psi_+^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]}, \tag{4}$$

and then by substituting this expression into (3) we get the following second-order equation:

$$\left(\pi_\mu\pi^\mu - m^2 - \frac{\mathrm{i}(k+1)}{2s}S_{\mu\nu}F^{\mu\nu}\right)\psi_+^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]} = 0. \tag{5}$$

Here $S_{\mu\nu}$ are the generators of the Lorentz group whose action on the antisymmetric tensor-spinor $\psi_+^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]}$ is given by the following formula:

$$\begin{aligned} S^{\mu\nu}\psi_+^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]} &= \frac{\mathrm{i}}{4}[\gamma^\mu,\gamma^\nu]\psi_+^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]} \\ &\quad + \mathrm{i}\Sigma_{\mathcal{P}}\left(g^{\mu\mu_1}\psi_+^{[\nu\nu_1][\nu_2\mu_2]\cdots[\nu_n\mu_n]} - g^{\nu\mu_1}\psi_+^{[\mu\nu_1][\nu_2\mu_2]\cdots[\nu_n\mu_n]}\right. \\ &\quad \left. - g^{\mu\nu_1}\psi_+^{[\nu\mu_1][\nu_2\mu_2]\cdots[\nu_n\mu_n]} + g^{\nu\nu_1}\psi_+^{[\mu\mu_1][\nu_2\mu_2]\cdots[\nu_n\mu_n]}\right), \end{aligned} \tag{6}$$

where $g^{\mu\nu}$ is the metric tensor with signature $(+,-,-,-)$.

In accordance with its definition tensor $\psi_+^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]}$ transforms according the representation $D(s-1/2,0)\otimes D(1/2,0)\oplus D(0,s-1/2)\otimes D(0,1/2)\equiv D(s,0)\oplus D(s-1,0)\oplus D(0,s)\oplus D(0,s-1)$ of the Lorentz group. Moreover, condition (2) reduces this representation to $D(s,0)\oplus D(0,s)$ whose generators without loss of generality can be expressed as

$$S_{ab} = \varepsilon_{abc}S_c, \qquad S_{0a} = \mathrm{i}\hat{\varepsilon}S_a, \qquad a,b = 1,2,3,$$

where $S_a$ are the matrices forming a direct sum of two irreducible representation $D(s)$ of algebra $so(3)$, $\varepsilon_{abc}$ is totally antisymmetric unit tensor of rank 3 and $\hat{\varepsilon}$ is an involutive matrix distinct from the unit one and commuting with $S_a$. This matrix can be expressed via the Casimir operators $C_1 = S_{\mu\nu}S^{\mu\nu}$ and $C_2 = \frac{1}{4}\varepsilon_{\mu\mu\lambda\sigma}S^{\mu\nu}S^{\lambda\sigma}$ of the Lorentz group

$$\hat{\varepsilon} = C_1C_2^{-1}. \tag{7}$$

Thus instead of (5) we can study the equivalent equation

$$\left(\pi_\mu\pi^\mu - m^2 - \frac{\mathrm{i}(k+1)}{2s}\hat{\varepsilon}S_a\left(\mathrm{i}F^{0a} + \frac{1}{2}\varepsilon_{0abc}F^{bc}\right)\right)\Psi = 0, \tag{8}$$

where $\Psi$ is a $2(2s+1)$-component spinor belonging to the space of irreducible representation $D(s,0)\oplus D(0,s)$ of the Lorentz group, $S_a(a=1,2,3)$ are direct sums of two $(2s+1)\times(2s+1)$ matrices forming the irreducible representation $D(s)$ of algebra $so(3)$. The components of the related tensor $\psi_+^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]}$ can be expressed via components of spinor $\Psi$ by using the Wigner coefficients (see section 7).

Taking into account commutativity of matrix $\hat{\varepsilon}$ with matrices $S_a$ equation (6) can be decoupled to two subsystems:

$$\left(\pi_\mu\pi^\mu - m^2 - \frac{\mathrm{i}(k+1)}{2s}\varepsilon S_a\left(\mathrm{i}F^{0a} + \frac{1}{2}\varepsilon_{0abc}F^{bc}\right)\right)\Psi_\varepsilon = 0, \tag{9}$$

where $\Psi_\varepsilon$ are eigenvectors of $\hat{\varepsilon}$ corresponding to the eigenvalues $\varepsilon = \pm 1$.

Thus in spite of a rather complicated form of the first-order equations (3) we received the second-order equation (5) for $\psi_+^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]}$ and relation (4) which expresses $\psi_-^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]}$ in terms of $\psi_+^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]}$. Moreover, equation (5) can be reduced to (9). This equation has been already solved for particle interacting with a constant and homogeneous magnetic field [9]. We shall see that equations (9) are very convenient and are easy to handle in the important case when the external field is generated by a point charge.

## 3. Radial equations for the Coulomb problem

Consider a charged particle with arbitrary half-integer spin $s$ and electric charge $e$ interacting with an external electromagnetic field. When this field is generated by a point charge $Ze$ the related vector-potential has the form

$$\mathbf{A} = 0, \qquad A_0 = \frac{\alpha}{r}, \tag{10}$$

where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ and $\alpha = Ze^2$.

Now we shall use the reduced version of the equations of motion given by expressions (5) to (9). Since both equations (9) corresponding to $\varepsilon = 1$ and to $\varepsilon = -1$ lead to the same energy spectrum we shall consider only the case $\varepsilon = 1$ and omit index $\varepsilon$ of the function $\Psi_\varepsilon$ in the formulae of this section which follow.

For the states with energy $E$ the corresponding solutions $\Psi$ can be written as

$$\Psi = \exp(-\mathrm{i}Ex_0)\psi(\mathbf{r}), \tag{11}$$

where $\psi(\mathbf{r})$ is a $(2s+1)$-component function depending on spatial variables and satisfying the following second-order equation:

$$\left(E - \frac{\alpha}{r}\right)^2 \psi = \left(m^2 - \Delta + \mathrm{i}k\alpha\frac{\mathbf{S}\cdot\mathbf{r}}{r^3}\right)\psi. \tag{12}$$

Taking into account the rotational invariance of equation (12) it is convenient to expand its solutions in terms of spherical spinors $\Omega^s_{j\,l\,m}$:

$$\psi = \xi_\lambda(r)\Omega^s_{j\;j-\lambda\,m}, \tag{13}$$

where $\Omega^s_{j\,l\,m}$ are orthonormalized joint eigenvectors of the following four commuting operators: of total angular momentum square $J^2$, orbital momentum square $L^2$, spin square $S^2$ and of the third component of the total angular momentum $J_3$, whose eigenvalues are $j(j+1), l(l+1), s(s+1)$ and $m$, respectively. Denoting $l = j - \lambda$ we receive

$$m = -j, -j+1, \ldots, j,$$

and

$$\lambda = -s, -s+1, \cdots - s + 2m_{sj},$$

where $m_{sj} = s$ if $s \leqslant j$ and $m_{sj} = j$ if $s > j$.

The expressions for spherical spinors via spherical functions are given in the appendix.

Thus we expand $\psi(\mathbf{x})$ in accordance with formula (13) where $\xi$ are radial functions and summation over $\lambda$ is imposed which takes the values indicated in the above. We note that the action of the scalar matrix $\mathbf{S}\cdot\mathbf{r}$ to the spinors $\Omega_{j\;j-\lambda\,m}$ is well defined and given by the formula [10]

$$\mathbf{S}\cdot\mathbf{r}\;\Omega_{j\;j-\lambda\,m} = rK^{sj}_{\lambda\lambda'}\Omega_{j\;j-\lambda' m}, \tag{14}$$

where $K^{sj}_{\lambda\lambda'}$ are numerical coefficients whose values are presented in the appendix.

Substituting (13) into (12), using (14) and the following representation for the Laplace operator $\Delta$:

$$\Delta = \frac{1}{r^2}\left(\frac{\partial}{\partial x}\left(r^2\frac{\partial}{\partial x}\right) - L^2\right), \tag{15}$$

where $L^2$ is the square of the orbital momentum operator $\mathbf{L} = \mathbf{r}\times\mathbf{p}$, we receive the following equations for the radial functions:

$$F\xi_\lambda = \frac{1}{r^2}M_{\lambda\lambda'}\xi_{\lambda'}, \tag{16}$$

where $F$ is the second-order differential operator

$$F = \frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r} + (E + \alpha/r)^2 - m^2 - \frac{j(j+1)}{r^2} \tag{17}$$

and $M$ is a matrix whose elements are

$$M_{\lambda\lambda'} = \lambda(\lambda - 2j - 1)\delta_{\lambda\lambda'} + \mathrm{i}g\alpha K^{sj}_{\lambda\lambda'}. \tag{18}$$

Formula (16) presents the equation for the radial wavefunction of a particle with arbitrary half-integer spin interacting with the Coulomb field.

## 4. Energy spectrum

Matrix $M$ is *normal*, i.e., it satisfies the condition $MM^\dagger = M^\dagger M$. Thus it is possible to diagonalize it using some invertible matrix $U$:

$$M \to \tilde{M} = UMU^{-1}, \qquad \tilde{M}_{\lambda\lambda'} = \delta_{\lambda\lambda'}\nu_\lambda, \tag{19}$$

(where $\nu_\lambda$ are eigenvalues of $M$) thus system (16) is reduced to the sequence of decoupled equations

$$F\tilde{\xi}^\lambda = \frac{1}{r^2}\nu_\lambda\tilde{\xi}^\lambda \qquad \text{(no sum over } \lambda), \tag{20}$$

where $\tilde{\xi}^\lambda$ is a $\lambda$ component of vector $\tilde{\xi} = U\xi$.

The other interpretation of $\tilde{\xi}^\lambda$ is as follows. Let $\xi^{(\lambda)}$ be an eigenvector of matrix $M$ corresponding to the eigenvalue $\nu_\lambda$. Then it can be represented as

$$\xi^{(\lambda)} = \tilde{\xi}^\lambda\omega_\lambda \qquad \text{(no sum over } \lambda), \tag{21}$$

where $\omega_\lambda$ is the eigenvector of $M$ which does not depend on $r$, and $\tilde{\xi}^\lambda$ is a multiplier depending on $r$.

Changing the variables $r \to \rho = 2\sqrt{m^2 - E^2}r$, $\tilde{\xi} \to f = \sqrt{\frac{\rho}{m^2-E^2}}\tilde{\xi}$ equation (20) is transformed to the well-known form

$$\rho\frac{\mathrm{d}^2 f}{\mathrm{d}\rho^2} + \frac{\mathrm{d}f}{\mathrm{d}\rho} + \left(\beta - \frac{\rho}{4} - \frac{k_\lambda^2}{4\rho}\right) f = 0, \tag{22}$$

where

$$\beta = \frac{\alpha E}{\sqrt{m^2 - E^2}}, \qquad k_\lambda^2 = (2j+1)^2 + 4\nu_\lambda - 4\alpha^2. \tag{23}$$

Note that equation (23) (but with another meaning of parameters $k_\lambda$) appears in the non-relativistic hydrogen system [11] and can be easily integrated. For the bound states, i.e., for $m^2 > E^2$, its solutions can be expressed via degenerated hypergeometric function $\mathcal{F}(\tilde{n}, d, \rho)$ as

$$f = C\rho^{\frac{k_\lambda}{2}} \exp\left(-\frac{\rho}{2}\right) \mathcal{F}\left(\frac{k_\lambda + 1}{2} - \beta, k_\lambda + 1, \rho\right), \tag{24}$$

where $C$ is an integration constant.

Solutions (24) are bounded at infinity provided the argument $\tilde{n} = \frac{k_\lambda+1}{2} - \beta$ is a non-positive integer, i.e., $\tilde{n} = -n' = 0, -1, -2, \ldots$. Then from (23) we obtain the possible values of energy for bound states:

$$E = m\left(1 + \frac{\alpha^2}{((n' + 1/2 + k_\lambda)^2 - \alpha^2)^{\frac{1}{2}}}\right)^{-\frac{1}{2}}. \tag{25}$$

Here $k_\lambda$ are parameters defined in expression (23), where $\nu_\lambda$ takes the values which coincide with the roots of the characteristic equation for matrix $M$:

$$\det(M - \nu_\lambda I) = 0, \tag{26}$$

where $I$ is the unit matrix of the appropriate dimension $D = 2s + 1$ for $s \leqslant j$ or $D = 2j + 1$ for $j \leqslant s$.

Thus we present the exact values of energy levels for the Coulomb system for the orbital particle having arbitrary half-integer spin. However, formulae (25), (23) include parameter $\nu_\lambda$ defined with the help of the algebraic equation (26) of order $D$ which can be solved in radicals for $j \leqslant 3/2$, or $s = 3/2$. For other values of $s$ and $j$ formula (26) defines an algebraic equation whose order is larger than 4, which does not have exact analytic solutions. The related possible values of $\nu_\lambda$ should be calculated numerically.

In the following section we find analytic expressions for approximate solutions of (26) and expand energy levels (25) in the power series of $(g\alpha)^2$.

## 5. General discussion of energy spectrum

First we note that if $g = 2$ and $s = 1/2$ then equation (25) reduces to the exact Sommerfeld formula for energy levels of the Dirac particle in the Coulomb field. Indeed, using (A.1) and solving (26) for $s = 1/2$, $g = 2$ we obtain

$$\nu_\lambda = \frac{1}{4} + 2\lambda\sqrt{\left(j + \frac{1}{2}\right)^2 - \alpha^2}, \qquad \lambda = \pm\frac{1}{2}$$

so that the related formula (25) takes the following form:

$$E = m\left(1 + \frac{\alpha^2}{\left(n' + \sqrt{\left(j + \frac{1}{2}\right)^2 - \alpha^2}\right)^2}\right)^{-1/2},$$

i.e., coincides with the Sommerfeld formula. This seems to be rather curious since the Dirac equation appears as a very particular case of equation (3) in which wavefunction $\psi^{[\mu_1\nu_1]\cdots[\mu_n\nu_n]}$ has zero number of pairs of anticommuting indices $[\mu_i\nu_i]$. The second-order equation (12), with $g = 2$ and $\mathbf{S}$ being matrices of spin 1/2 , appears in the Dirac theory too if we use potential (10) and express the 'small' components of the wavefunction in terms of 'large' ones (this procedure is equivalent to our substitution (4)). Thus the spectrum of energies for higher spin fermions (25) includes the spectrum of the Dirac particle as a particular case.

In order to analyse spectrum of energies of the orbital fermion with arbitrary spin we shall look for approximate solutions of equation (26). Considering $\alpha$ as a small parameter, using explicit expressions (18) and (A.1) for matrix $M$ and applying the standard perturbation technique, we obtain analytic expressions for approximate solutions of this equation for arbitrary spin and total angular momenta:

$$\nu_\lambda = \lambda^2 - (2j+1)\lambda + \frac{(\alpha g)^2}{8}\left(\frac{(\lambda+s)(2j-\lambda-s+1)(s-\lambda+1)(2j+s-\lambda+2)}{(2j-2\lambda+1)(2j-2\lambda+3)(j-\lambda+1)}\right.$$
$$\left. - \frac{(s+\lambda+1)(s-\lambda)(2j-s-\lambda)(2j+s-\lambda+1)}{(j-\lambda)(2j-2\lambda+1)(2j-2\lambda-1)}\right) + O(\alpha^4). \tag{27}$$

Starting with (25) and using (27) we find approximate expressions for energy levels up to the terms of order $\alpha^4$:

$$\frac{E}{m} = 1 - \frac{\alpha^2}{2n^2} + \frac{3\alpha^4}{8n^4} - \frac{\alpha^4}{n^3(2l+1)} + \frac{g^2\alpha^4}{8n^3(2l+1)^2}\left(\frac{(j-l+s)(j+l-s+1)(l-j+s+1)(j+l+s+2)}{(l+1)(2l+3)} - \frac{(j-l+s+1)(j+l-s)(l-j+s)(j+l+s+1)(1-\delta_{l0})}{l(2l-1)}\right), \qquad (28)$$

where $n$ and $l$ are the quantum numbers which take the values $n = n' + l + 1 = 1, 2, \ldots, l = 0, 1, \ldots, n-1$.

Formula (28) defines the fine structure of the energy spectrum for particle with arbitrary spin. We see that energy levels are labelled by three quantum numbers, i.e., by $n$, $l$ and $j$. Any level of order $\alpha^2$ corresponding to the main quantum number $n$ is split to $N_n$ sublevels which have the order $\alpha^4$ and describe the fine structure of the spectrum. The value of $N_n$ coincides with the number of possible pairs $(l, j)$, i.e.,

$$N_n = \begin{cases} n^2 & \text{for } n \leqslant s, \\ \left(s+\frac{1}{2}\right)\left(2n - s - \frac{1}{2}\right) & \text{for } n > s. \end{cases}$$

We recall that for the Dirac particle the number of split sublevels is equal to $n$ and any level with $j \neq n - 1/2$ is double degenerated. This again is in accordance with our formula (28) if we put $s = 1/2$ and $g = 2$ and find the related approximate $\nu_\lambda$ from (27).

The physical interpretation of spectrum (28) is obtained by using the results of paper [1] where the Foldy–Wouthuysen reduction of equations (23) was carried out. In addition to the rest energy term proportional to $m$ formula (28) includes the non-relativistic Balmer term $-\frac{\alpha^2}{2n^2}$ and the term $\alpha^4\left(\frac{3}{8n^4} - \frac{1}{n^3(2l+1)}\right)$ which represents the relativistic correction to the kinetic energy. The last terms in (28), which are proportional to $g^2\alpha^4$, represent the contribution caused by the spin–orbital, Darwin and quadrupole interactions.

## 6. Energy levels for $s = 3/2$

In the previous section we have defined the energy levels for a charged particle with arbitrary half-integer spin interacting with the Coulomb potential and briefly described the related wavefunctions. Here we consider in full detail the case $s = 3/2$.

First we note that in the case $s = 3/2$ it is possible to solve the corresponding algebraic equation (26) in radicals and to find the associated exact formulae (25). We shall not present these rather cumbersome formulae here and restrict ourselves to analysis of approximate solutions (28).

The energy levels (28) for $s = 3/2$ are reduced to the following form:

$$E = m\left(1 - \frac{\alpha^2}{2n^2} + \frac{3\alpha^4}{8n^4} + \frac{\alpha^4}{n^3}\Delta_{j\,l}\right). \qquad (29)$$

Here $j$ is the total angular moment quantum number which can take the following values:

$$j = \begin{cases} \frac{3}{2}, & \text{if } l = 0, \\ \frac{5}{2},\ \frac{3}{2},\ \frac{1}{2}, & \text{if } l = 1, \\ l+\frac{3}{2}, l+\frac{1}{2},\ l-\frac{1}{2}, l+\frac{3}{2}, & \text{if } l > 1, \end{cases} \qquad (30)$$

and $\Delta_{j\,l}$ are corrections to energy levels defined by $j$ and $l$:

$$\Delta_{j\,l} = \frac{g^2}{8(2l+1)^2}\left(\frac{\left(j-l+\frac{3}{2}\right)\left(j+l-\frac{1}{2}\right)\left(l-j+\frac{5}{2}\right)\left(j+l+\frac{7}{2}\right)}{(l+1)(2l+3)} - \frac{\left(j-l+\frac{5}{2}\right)\left(j+l-\frac{3}{2}\right)\left(l-j+\frac{3}{2}\right)\left(j+l+\frac{5}{2}\right)(1-\delta_{l0})}{l(2l-1)}\right) - \frac{1}{2l+1}. \tag{31}$$

Thus in accordance with (30), (31) we have

$$\begin{aligned}
\Delta_{l+3/2\,l} &= \frac{3g^2(2l+5)}{8(l+1)(2l+1)(2l+3)} - \frac{1}{2l+1} && \text{for any } l,\\
\Delta_{l+1/2\,l} &= \frac{g^2(2l^2-9l-27)}{8l(l+1)(2l+1)(2l+3)} - \frac{1}{2l+1} && \text{for } l>0,\\
\Delta_{l-1/2\,l} &= \frac{g^2(16-2l^2-13l)}{8l(l+1)(2l+1)(2l-1)} - \frac{1}{2l+1} && \text{for } l>0,\\
\Delta_{l-3/2\,l} &= \frac{3g^2(3-2l)}{8l(2l+1)(2l-1)} - \frac{1}{2l+1} && \text{for } l>1.
\end{aligned} \tag{32}$$

Formulae (29)–(32) describe the energy spectrum of a particle of spin 3/2 interacting with the field of a point charge up to order $\alpha^4$. The first of relations (32) is well defined for any $l$ while the others take place for $l > 0$ or $l > 1$ only.

We see that if $l > 1$ then any level corresponding to fixed values of the quantum numbers $n$ and $l$ is split to four sublevels representing a fine structure of the spectrum. The value of the fine splitting decreases as the quantum numbers $n$ and $l$ increase.

We note that unlike the case $s = 1/2$ energy levels (29) are in general non-degenerate. The fine structure of spectrum (29)–(32), however, includes an arbitrary parameter $g$ which can be interpreted as a gyromagnetic ratio of the system [1]. This parameter can be fixed by means of experimental or theoretical requirements and some degeneracy can appear for its special values.

There are three 'privileged' values of $g$, namely the generally recognized value $g = 2$ (which is predicted by string theory and is in accordance with experimental data [12]), $g = 1/s$ (which naturally appears in relativistic models without anomalous interaction [13]) and $g = \sqrt{2/s}$. For $g \neq \sqrt{2/s}$ it is possible to show that the spectrum (29)–(32) is degeneracy free in general and for $g = 2$ or $g = 1/s$ even accidental degeneracy free. However, for $g = \sqrt{2/s}$ this spectrum is doubly degenerate. Moreover, as we shall see, the origin of this degeneracy is similar to that observed in the Dirac–Coulomb system (cf [14]).

Setting in (32) $g = \sqrt{2/s}$ we obtain for $s = 3/2$:

$$\begin{aligned}
\Delta_{l+\frac{3}{2}\,l} &= -\frac{4l(l+2)+1}{2(2l+1)(l+1)(2l+3)} && \text{for any } l,\\
\Delta_{l+\frac{1}{2}\,l} &= -\frac{4l^2(3l+7)+27(l+1)}{6l(l+1)(2l+1)(2l+3)}, && l>0,\\
\Delta_{l-\frac{1}{2}\,l} &= -\frac{4l^2(3l+2)+7l-16}{6l(2l-1)(2l+1)(l+1)}, && l>0,\\
\Delta_{l-\frac{3}{2}\,l} &= -\frac{4l^2-3}{2l(2l+1)(2l-1)}, && l>1.
\end{aligned} \tag{33}$$

Comparing $\Delta_{l+\frac{3}{2}\,l}$ with $\Delta_{l-\frac{3}{2}\,l}$ in (33) it is not difficult to observe a double degeneracy of the related energy levels, due to the fact that

$$\Delta_{l+\frac{3}{2}\,l} = \Delta_{(l+1)-\frac{3}{2}(l+1)}, \qquad l \neq 0. \tag{34}$$

Thus if the gyromagnetic ratio is equal to $\sqrt{\frac{2}{s}}$ then like in the Dirac–Coulomb system the energy levels corresponding to maximal or minimal possible value of $j$ for given $l > 0$ are doubly degenerate. However, in some ways, this analogy is rather matter of convention since in the Dirac–Coulomb system all energy values (except the ground one) are doubly degenerate. In our system with spin 3/2 like in the Dirac–Coulomb one there are degenerate states with $j = l \pm s$. But in addition for $s = 3/2$ there are states with $j = l \pm (s-1)$ and with $l = 0$ which are singlets.

We note that analysis of energy spectrum (28) presented in this section admits a straightforward extension to the case of any spin. In particular, our conclusion concerning degeneracy of the spectrum for special value $g = \sqrt{\frac{2}{s}}$ of the gyromagnetic ratio is valid for any $s$.

## 7. Explicit solutions of the relativistic wave equation for $s = \frac{3}{2}$

Following the procedure outlined in sections 4 and 5 it is not difficult to find the explicit expressions for the wavefunction of the particle with spin 3/2 in the Coulomb field.

We start with the basic equation (3). In the case $s = 3/2$ the related wavefunction is a tensor-bispinor $\psi^{[\mu\nu]}$ which has 24 components, and equation (3) is reduced to the following form:

$$(\gamma_\lambda \pi^\lambda - m)\psi^{[\mu\nu]} - \frac{1}{6}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\pi_\lambda\gamma_\sigma\psi^{[\lambda\sigma]} + \frac{4\mathrm{i}ek}{3m}\left(\frac{1}{4}\gamma_\alpha\gamma_\sigma F^{\alpha\sigma}\psi_+^{[\mu\nu]} + F_\alpha{}^\mu\psi_+^{[\nu\alpha]} - F_\alpha{}^\nu\psi_+^{[\mu\alpha]}\right) = 0, \tag{35}$$

where (and in the following)

$$\psi_\pm^{[\mu\nu]} = \psi^{[\mu\nu]} \mp \tfrac{1}{2}\gamma_5\varepsilon^{\mu\nu}{}_{\lambda\sigma}\psi^{[\lambda\sigma]}. \tag{36}$$

Solving RWE (35) for $\psi_-^{[\mu\nu]}$ we obtain equations (5) and (4) for particle with spin 3/2:

$$\left(\pi_\alpha\pi^\alpha - m^2 - \frac{g}{2}S_{\alpha\sigma}F^{\alpha\sigma}\right)\psi_+^{[\mu\nu]} = 0, \tag{37}$$

$$\psi_-^{[\mu\nu]} = \frac{1}{m}\gamma_\lambda\pi^\lambda\psi_+^{[\mu\nu]}. \tag{38}$$

Functions $\psi_+^{[\mu\nu]}$ and $\psi_-^{[\mu\nu]}$ both have 12 independent components and form carrier spaces for representations $D(3/2,0)\oplus D(0,3/2)\oplus D(1/2,0)\oplus D(0,1/2)$ and $D(1,1/2)\oplus D(1/2,1)$ of the Lorentz group respectively.

In addition, $\psi^{[\mu\nu]}$ has to satisfy the condition (2):

$$\gamma_\mu\gamma_\nu\psi^{[\mu\nu]} = 0. \tag{39}$$

In accordance with its definition, tensor $\psi_-^{[\mu\nu]}$ automatically satisfies the condition $\gamma_\mu\gamma_\nu\psi_-^{[\mu\nu]} = 0$ (even when condition (39) for $\psi^{[\mu\nu]}$ is not imposed). Function $\psi_+^{[\mu\nu]}$ satisfies the condition $\gamma_\mu\gamma_\nu\psi_+^{[\mu\nu]} = 0$ as well-provided equation (39) is satisfied. Moreover, relation (39) reduces the number of independent components of $\psi_+^{[\mu\nu]}$ to eight which form a carrier space of the representation $D(3/2,0)\oplus D(0,3/2)$.

The action of the generators of the Lorentz group on $\psi_+^{[\lambda\sigma]}$ can be derived from equation (6) for $n = 1$:

$$S^{\mu\nu}\psi_+^{[\lambda\sigma]} = \frac{\mathrm{i}}{4}[\gamma^\mu,\gamma^\nu]\psi_+^{[\lambda\sigma]} + \mathrm{i}\left(g^{\mu\lambda}\psi_+^{[\nu\sigma]} - g^{\nu\lambda}\psi_+^{[\mu\sigma]} - g^{\mu\sigma}\psi_+^{[\nu\lambda]} + g^{\nu\sigma}\psi_+^{[\mu\lambda]}\right). \tag{40}$$

In fact the same action is valid for any tensor-bispinor, e.g., for $\psi^{[\lambda\sigma]}$ and $\psi_-^{[\lambda\sigma]}$. For $\psi_+^{[\lambda\sigma]}$ formula (40) can be specified taking into account equations (36) and (39). First we note that in accordance with (36)

$$\psi_+^{[0a]} = \tfrac{1}{2}\varepsilon^{0a}{}_{bc}\gamma_5\psi_+^{[bc]} \qquad (a, b, c = 1, 2, 3) \tag{41}$$

so that we can restrict ourselves to spatial components $\psi_+^{[ab]}$ of the tensor-spinor with $a, b = 1, 2, 3$ (we denote superindices running from 1 to 3 by Latin letters). Denoting

$$\psi_+^{[ab]} = \varepsilon^{abc}\eta^c \tag{42}$$

and using relations (36), (39), (40) and (41) we obtain the action of generators of the Lorentz group in the following form:

$$S^{ab}\eta^c = \varepsilon^{abk}S^k\eta^c, \qquad S^{0a}\eta^c = \mathrm{i}\gamma_5 S^a\eta^c, \tag{43}$$

where

$$S_a = S_a^{(\frac{1}{2})} + S_a^{(1)}, \qquad S_a^{(\frac{1}{2})} = \tfrac{\mathrm{i}}{4}\varepsilon_{abc}\gamma_b\gamma_c \tag{44}$$

and $S_a^{(1)}$ are spin one matrices whose elements are $\left(S_a^{(1)}\right)_{bc} = \mathrm{i}\varepsilon_{abc}$.

We note that like tensor-spinor $\psi_+^{[\mu\nu]}$, vector-spinor $\eta^a \equiv \eta^a_\alpha$ has an implicit spinorial index $\alpha$ which we omit. Moreover, $\gamma$ matrices act on the spinorial index of $\eta^a$ while matrices $S_a^{(1)}$ act on the vector index $a$ so that $S_a^{(1)}$ commute with $\gamma_\mu$ by definition. Thus matrices $S_a$ in (44) are sums of commuting spin 1/2 and spin 1 matrices, and so that they can be reduced to the direct sums of spin 3/2 and spin 1/2 matrices. Such reduction is easily handled by means of Wigner coefficients provided the corresponding third components of spin vectors are diagonal.

If we choose the following realization of the Dirac matrices:

$$\gamma_0 = \begin{pmatrix}\hat{0} & I\\ I & \hat{0}\end{pmatrix}, \qquad \gamma_a = \begin{pmatrix}\hat{0} & -\sigma_a\\ \sigma_a & \hat{0}\end{pmatrix}, \qquad \mathrm{i}\gamma_5 = \begin{pmatrix}I & \hat{0}\\ \hat{0} & -I\end{pmatrix}, \tag{45}$$

where $I$ and $\hat{0}$ are the $2\times 2$ unit and zero matrices, respectively, and $\sigma_a$ are the Pauli matrices, then both $\mathrm{i}\gamma_5$ and $S_3^{(\frac{1}{2})}$ are diagonal. Using the unitary transformation $\eta^a \to W^{ab}\eta^b$ with the transformation matrix

$$W = \frac{1}{\sqrt{2}}\begin{pmatrix}-\mathrm{i} & 0 & \mathrm{i}\\ 1 & 0 & 1\\ 0 & \mathrm{i}\sqrt{2} & 0\end{pmatrix} \tag{46}$$

we diagonalize $S_3^{(1)}$ too since in this case

$$S_3^{(1)} \to \tilde{S}_3^{(1)} = WS_3^{(1)}W^\dagger = \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}.$$

Thus we pass from (44) to the following realization of spin matrices:

$$S_a \to \hat{S}_a = WS_aW^\dagger = \tfrac{1}{2}\sigma_a + \tilde{S}_a^{(1)}. \tag{47}$$

Now we are ready to apply the Wigner coefficients $C^{sm}_{s_1m_1s_2m_2}$ to reduce (47) to a direct sum of spin 3/2 and spin 1/2 matrices. In our case $s_1 = 1, s_2 = 1/2, s = 3/2, 1/2$ and the following notation is used: $m = k,\ m_1 = c, m_2 = \alpha$. Then the basis $\Psi^{sk}$ which corresponds to the completely reduced representation of spin matrices is connected with $\eta^a_\alpha$ by means of the formula

$$\Psi^{sk} = C^{sk}_{1c\frac{1}{2}\alpha}W_{ca}\eta^a_\alpha. \tag{48}$$

Condition (2) suppresses the states with $s = 1/2$ so that $\Psi^{\frac{1}{2}k} = 0$. Then, denoting $\Psi^{\frac{3}{2}k} = \Psi^k$ and using (48) we obtain

$$\Psi^k = C^{\frac{3}{2}k}_{1c\frac{1}{2}\alpha} W_{ca}\eta^a_\alpha. \tag{49}$$

Thus relations (41), (42) and (49) explicitly show how functions $\psi_+^{[\mu\nu]}$ (which satisfy equations (37)) are connected with solutions $\Psi^k$ of the reduced equation (8). The inverse relations have the form

$$\psi^{[0c]}_{\alpha+} = (\gamma_5)_{\alpha\sigma}\left(W^\dagger\right)_{ck} C^{\frac{3}{2}n}_{1k\frac{1}{2}\sigma}\Psi^n, \qquad \psi^{[ab]}_{\alpha+} = -\gamma_5\varepsilon_{abc}\psi^{[0c]}_{\alpha+}. \tag{50}$$

In accordance with expressions (19)–(24) solutions of the reduced equations (8) corresponding to energy $E = E_{n'j\nu_\lambda}$ (25) can be written as

$$\Psi^k_{n'j\nu_\lambda} = \exp(-\mathrm{i}E_{n'j\ \nu_\lambda}x_0)\omega_\lambda\tilde{\xi}_\lambda\Omega^{\frac{3}{2}}_{j\ j-\lambda\, m}, \tag{51}$$

where $\omega_\lambda$ are the eigenvectors of matrix $M$ whose elements $M_{\lambda\lambda'}$ are given by formula (18) (see (appendix)) and

$$\tilde{\xi}_\lambda = C_\lambda\left(m^2 - E^2_{n'j\ \nu_\lambda}\right)^{\frac{k_\lambda+1}{4}} r^{\frac{k_\lambda-1}{2}} \mathrm{e}^{\left(m^2-E^2_{n'j\nu_\lambda}\right)r}\mathcal{F}\left(-n', k_\lambda+1, 2\left(m^2 - E^2_{n'j\nu_\lambda}\right)^{\frac{1}{2}}r\right). \tag{52}$$

Substituting $\Psi^k = \Psi^k_{n'j\,\nu_\lambda}$ from (51) into (50) we obtain the explicit expressions for (non-normalized) solutions of equation (35).

To complete presentation of solutions of the reduced equation (8) for spin $s = 3/2$ we present the explicit expressions of spherical spinors $\Omega^{\frac{3}{2}}_{j\ j-\lambda\, m}$ in the appendix.

## 8. Discussion

In this paper the explicit solution of the relativistic quantum mechanical Kepler problem for orbital particle with arbitrary half-integer spin are presented. As a mathematical model of such a system we use the tensor-spinor relativistic wave equations (RWE) proposed in paper [1].

Tensor-spinor formulation of RWE makes it possible to overcome the main fundamental difficulties which appears in relativistic theory of particles with higher spins such as causality violation and complex energies predicted for particles interacting with the constant magnetic field [1]. In addition, this formulation is rather straightforward and simple which makes it possible to extend the known exact solutions for some special quantum mechanical problems described by the Dirac equation [2] to the case of an arbitrary half-integer spin. The problem of interaction of arbitrary spin fermions with constant and homogeneous magnetic field was solved in paper [9].

In the present paper we solve a much more complicated problem: interaction of arbitrary spin fermion with the field of a point charge. In addition, tensor-spinor RWE admit exact solutions for the cases of interaction with the plane wave field, with constant crossed electric and magnetic fields and some others. We plan to present solutions of the related problems in the future.

In addition to the general treatment for an arbitrary spin we restrict in full details to case $s = 3/2$. However, the results given in sections 6 and 7 admit straightforward generalization to the case of $s$ arbitrary. In particular, in analogy with (41)–(50) the tensor-spinor $\psi_+^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]}$ with arbitrary $n$ in view of equation (2) is effectively reduced to symmetric tensor $\eta^{\mu_1\mu_2\cdots\mu_n}$ with $n$ indices running from 1 to 3 and then to a $2(2s + 1)$-component function $\psi$ satisfying equation (12).

We obtain the generalized Sommerfeld formula (25) for energy levels of particle with an arbitrary half-integer spin interacting with the Coulomb potential. In contrast to the formula generated by the Dirac equation our expression (25) includes parameter $g$ whose value is not fixed *a priori*. In accordance with the analysis present in [1] this parameter is associated with the gyromagnetic ratio of a particle described by equations (2), (3).

Analysing formula (28) for energy levels we conclude that in addition to the most popular values $g = 1/s$ and $g = 2$ there exists one more intriguing value, namely $g = \sqrt{2/s}$ which corresponds to a specific degeneracy of the related energy spectrum. We note that in the case $s = 1/2$ all mentioned privileged values of the gyromagnetic ratio coincide while for $s > 1/2$ the relation $\frac{1}{s} < \sqrt{\frac{2}{s}} < 2$ is valid. In other words the degeneracy related value of $g$ lies between the recognized values $1/s$ and 2.

Notice that in contrast with the Procá theory [3, 4] the effective potential appearing in radial equations (12) does not include singular terms proportional to the Dirac $\delta$-function which can lead to the known inconsistencies with indefinite charge of the orbital particle.

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## Appendix

Here we present some technical data used in the main text.

Spherical spinors are eigenvectors of the commuting operators of squared total angular momentum $\mathbf{J}^2$, squared orbital momentum $\mathbf{L}^2$, squared spin $\mathbf{S}^2$ and the third components $J_3$ of the total angular momentum, so that

$$\begin{aligned} \mathbf{J}^2\Omega^s_{j\,l\,m} &= j(j+1)\Omega^s_{j\,l\,m}, \qquad & \mathbf{L}^2\Omega^s_{j\,l\,m} &= l(l+1)\Omega^s_{j\,l\,m}, \\ \mathbf{S}^2\Omega^s_{j\,l\,m} &= s(s+1)\Omega^s_{j\,l\,m}, \qquad & J_3\Omega^s_{j\,l\,m} &= m\Omega^s_{j\,l\,m}. \end{aligned}$$

Spherical spinors can be expressed via spherical functions $Y_{lm}$:

$$\left(\Omega^s_{j\,l\,m}\right)_\mu = C^{jm}_{lm-\mu s\ \mu} Y_{l\ m-\mu},$$

where $C^{jm}_{l\ m-\mu\ s\mu}$ are the Wigner coefficients, $\varphi$ and $\theta$ are the polar and the azimuthal angles of $\mathbf{r}$.

In particular, for $s = 3/2$, the spherical spinors can be represented as the following four-component rows:

$$\Omega^{\frac{3}{2}}_{j\ j-\frac{3}{2}\ m} = \frac{1}{2\sqrt{j(j-1)(2j-1)}}\begin{pmatrix} \sqrt{(j+m)(j+m-1)(j+m-2)}Y_{j-\frac{3}{2}\ m-\frac{3}{2}} \\ \sqrt{3(j^2-m^2)(j+m-1)}Y_{j-\frac{3}{2}\ m-\frac{1}{2}} \\ \sqrt{3(j^2-m^2)(j-m-1)}Y_{j-\frac{3}{2}\ m+\frac{1}{2}} \\ \sqrt{(j-m)(j-m-1)(j-m-2)}Y_{j-\frac{3}{2}\ m+\frac{3}{2}} \end{pmatrix},$$

$$\Omega^{\frac{3}{2}}_{j\ j-\frac{1}{2} m} = \frac{1}{2\sqrt{j(j+1)(2j-1)}}\begin{pmatrix} -\sqrt{3(j+m)(j+m-1)(j-m+1)}Y_{j-\frac{1}{2}\ m-\frac{3}{2}} \\ -(j+1-3m)\sqrt{j+m}Y_{j-\frac{1}{2}\ m-\frac{1}{2}} \\ (j+1+3m)\sqrt{j-m}Y_{j-\frac{1}{2}\ m+\frac{1}{2}} \\ \sqrt{3(j-m)(j-m-1)(j+m-1)}Y_{j-\frac{1}{2}\ m+\frac{3}{2}} \end{pmatrix},$$

$$\Omega^{\frac{3}{2}}_{j\ j+\frac{1}{2}m} = \frac{1}{2\sqrt{j(j+1)(2j+3)}} \begin{pmatrix} \sqrt{3(j+m)(j-m+2)(j-m+1)}Y_{j+\frac{1}{2}\ m-\frac{3}{2}} \\ -(j+3m)\sqrt{j-m+1}Y_{j+\frac{1}{2}\ m-\frac{1}{2}} \\ -(j-3m)\sqrt{j+m+1}Y_{j+\frac{1}{2}\ m+\frac{1}{2}} \\ \sqrt{3(j-m)(j+m+1)(j+m+2)}Y_{j+\frac{1}{2}\ m+\frac{3}{2}} \end{pmatrix},$$

$$\Omega^{\frac{3}{2}}_{j\ j+\frac{3}{2}m} = \frac{1}{2\sqrt{(j+1)(j+2)(2j+3)}} \begin{pmatrix} -\sqrt{(j-m+1)(j-m+2)(j-m+3)}Y_{j+\frac{3}{2}\ m-\frac{3}{2}} \\ \sqrt{3(j+m+1)(j-m+1)(j-m+2)}Y_{j+\frac{3}{2}\ m-\frac{1}{2}} \\ -\sqrt{3(j-m+1)(j+m+1)(j+m+2)}Y_{j+\frac{3}{2}\ m+\frac{1}{2}} \\ \sqrt{(j+m+1)(j+m+2)(j+m+3)}Y_{j+\frac{3}{2}\ m+\frac{3}{2}} \end{pmatrix}.$$

The result of action of matrix $\mathbf{S}\cdot\mathbf{r}$ to spherical spinors is given by relation (14) with [10, 13]

$$K^{sj}_{\lambda\lambda'} = -1/2\left(\delta_{\lambda'\lambda+1}\, a^{sj}_{s+\lambda} + \delta_{\lambda'\lambda-1}\, a^{sj}_{s+\lambda+1}\right), \tag{A.1}$$

where

$$a^{sj}_{\mu} = \left(\frac{\mu(2j+1-\mu)(2s+1-\mu)(2j+2s-\mu+2)}{(2s+2j-2\mu+1)(2s+2j-2\mu+3)}\right)^{1/2}.$$

For $s = 3/2$ coefficients (A.1) are reduced to the following form:

$$a^{\frac{3}{2}j}_{1} = \sqrt{\frac{3(j+1)}{j}}, \qquad a^{\frac{3}{2}j}_{2} = \sqrt{\frac{(2j-1)(2j+3)}{j(j+1)}}, \qquad a^{\frac{3}{2}j}_{3} = \sqrt{\frac{3j}{j+1}}. \tag{A.2}$$

The related matrix $M$ whose elements $M_{\lambda\lambda'}$ are given by formula (18) takes the form

$$M = \begin{pmatrix} \frac{15}{4}+3j & \frac{\mathrm{i}g\alpha}{2}\sqrt{\frac{3j}{j+1}} & 0 & 0 \\ \frac{\mathrm{i}g\alpha}{2}\sqrt{\frac{3j}{j+1}} & \frac{3}{4}+j & \frac{\mathrm{i}g\alpha}{2}\sqrt{\frac{(2j-1)(2j+3)}{j(j+1)}} & 0 \\ 0 & \frac{\mathrm{i}g\alpha}{2}\sqrt{\frac{(2j-1)(2j+3)}{j(j+1)}} & -\frac{1}{4}-j & \frac{\mathrm{i}g\alpha}{2}\sqrt{\frac{3(j+1)}{j}} \\ 0 & 0 & \frac{\mathrm{i}g\alpha}{2}\sqrt{\frac{3(j+1)}{j}} & \frac{3}{4}-3j \end{pmatrix}. \tag{A.3}$$

Eigenvalues $\nu$ of matrix (A.3) coincide with the roots of the characteristic equation for matrix (A.3) which is an algebraic equation of order 4:

$$\begin{aligned} 2\nu^4 - 10\nu^3 + \left(\tfrac{39}{4} + 5(g\alpha)^2 - 10j(j+1)\right)\nu^2 \\ + \left(18j(j+1) - \tfrac{9}{8} - \tfrac{57}{2}(g\alpha)^2\right)\nu + 18j^2(j+1)^2 \\ - \tfrac{9}{4}j(j+1) - \tfrac{5}{2}\left(\tfrac{3}{4}\right)^4 - (9j(j+1) - 48.75)(g\alpha)^2 = 0. \end{aligned} \tag{A.4}$$

The related eigenvectors $\omega_\lambda$ corresponding to eigenvalues $\nu_\lambda$ can be observed in the form of a four-component column whose components are

$$(\omega_\lambda)_1 = \frac{(g\alpha)^2\sqrt{3j(2j-1)(2j+3)}}{j+1}(4\nu_\lambda + 12j - 3),$$

$$(\omega_\lambda)_2 = \frac{\mathrm{i}(g\alpha)^2\sqrt{(2j-1)(2j+3)}}{2\sqrt{j(j+1)}}(15+12j-4\nu_\lambda)(3-12j-4\nu_\lambda),$$

$$(\omega_\lambda)_3 = \left((15+12j-4\nu_\lambda)(3+4j-4\nu_\lambda) + 12\frac{j+1}{j}(g\alpha)^2\right)(4\nu_\lambda + 12j - 3),$$

$$(\omega_\lambda)_4 = \frac{\mathrm{i}g\alpha\sqrt{3(j+1)}}{2\sqrt{j}}\left((15+12j-4\nu_\lambda)(3+4j-4\nu_\lambda) + 12\frac{j+1}{j}(g\alpha)^2\right).$$

Solving algebraic equation (A.4) it is possible to find the exact values of $\nu_\lambda$. We shall not present the related cumbersome formulae here whose expansion in power series of $(g\alpha)^2$ is given in formula (27).

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